Solvable rational extension of translationally shape invariant potentials

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Combining recent results on rational solutions to the Riccati-Schrödinger equations for shape invariant potentials to the scheme developed by Fellows and Smith in the case of the one-dimensional harmonic oscillator, we show that it is possible to generate an infinite set of solvable rational extensions for every translationally shape invariant potential of second category.

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I. INTRODUCTION

In quantum mechanics there exists only few families of potentials which are exactly solvable in closed-form. Most of them belong to the class of shape-invariant potentials [1–3]. A possible way to generate new solvable potentials is to start from the known ones and to construct regular rational extensions of them. If the procedure has a long history, in the last years important progress have been made in this direction [4–6]. A nice example of such a rational extension is provided by the so-called CPRS potential [7] which is a rational extension of the one-dimensional harmonic oscillator. Very recently Fellows and Smith [8] showed that this potential can be obtained as a supersymmetric partner of the harmonic oscillator. In the same way they show how to generate an infinite family of partner potentials which are regular rational extensions of the harmonic oscillator. This partnership is based on the use of excited states Riccati-Schrödinger (RS) functions as superpotentials. This technique was devised for the first time by Robnik [9, 10], but the potentials obtained are singular. Fellows and Smith circumvent the problem by using a "spatial Wick rotation" which eliminates the singularities from the real axis. In a recent work [11] we propose a general scheme to obtain rational solutions to the Riccati-Schrödinger equations associated to the whole class of translationally shape invariant potentials. These last are shared into two categories which are related via simple changes of variables respectively to the harmonic oscillator and to the isotonic oscillator. In this letter we show how, by combining these results with the Robnik-Fellows-Smith technique, we can generate an infinite set of regular rationally-extended solvable potentials from every shape invariant potential of the second category.

II. HARMONIC AND ISOTONIC OSCILLATORS

A. Basic scheme

Let $H = -d^2/dx^2 + V(x)$ of associated spectrum $(E_n, w_n) \equiv (E_n, \psi_n)$, where $w_n(x) = -\psi'_n(x)/\psi_n(x)$.
The Riccati-Schrödinger (RS) equation [11] for the level E_n is:

$$-w'_n(x) + w_n^2(x) = V(x) - E_n, \tag{1}$$

where we suppose $E_0 = 0$.
Make a "spatial Wick rotation", that is, set $x \to ix$, and define $v_n(x) = -iw_n(ix)$ Eq(1) becomes:

$$v'_n(x) + v_n^2(x) = V^{(n)}(x), \tag{2}$$

where:

$$V^{(n)}(x) = E_n - V(ix). \tag{3}$$

$V^{(n)}(x)$ is supposed to be real and to have no movable (that is n dependent) singularity on the real line. Considering $v_n(x)$ as superpotential, $V^{(n)}$ can be viewed as the SUSY partner[1] of $\widetilde{V}^{(n)}$ defined as:

$$\widetilde{V}^{(n)}(x) = V^{(n)}(x) - 2v_n'(x) = 2v_n^2(x) - V^{(n)}(x), \tag{4}$$

that is:

$$\widetilde{V}^{(n)}(x) = V(ix) - E_n + 2v_n^2(x). \tag{5}$$

The positive hamiltonians $H^{(n)}$ and $\widetilde{H}^{(n)}$, associated to $V^{(n)}(x)$ and $\widetilde{V}^{(n)}(x)$ respectively, can be written:

$$\begin{cases} \widetilde{H}^{(n)} = A^{(n)+}A^{(n)} \\ H^{(n)} = A^{(n)}A^{(n)+}, \end{cases} \tag{6}$$

where:

$$A^{(n)} = \frac{d}{dx} + v_n(x). \tag{7}$$

If $\widetilde{\psi}_0^{(n)}(x) \sim \exp\left(-\int v_n(x)dx\right)$ is normalizable, it satisfies $A^{(n)}\widetilde{\psi}_0^{(n)} = 0$ and is then the zero-energy ground state of $\widetilde{H}^{(n)}$.

In this case the two hamiltonians are almost isospectral, that is:

$$\begin{cases} \widetilde{E}_0^{(n)} = 0 \\ E_k^{(n)} = \widetilde{E}_{k+1}^{(n)}, \ k \geq 0, \end{cases} \tag{8}$$

and their eigenstates are related by:

$$\begin{cases} \psi_k^{(n)}(x) = \frac{1}{\sqrt{\widetilde{E}_{k+1}^{(n)}}} A^{(n)}\widetilde{\psi}_{k+1}^{(n)}(x) \\ \widetilde{\psi}_k^{(n)}(x) = \frac{1}{\sqrt{E_k^{(n)}}} A^{(n)+}\psi_k^{(n)}(x). \end{cases} \tag{9}$$

If $\widetilde{\psi}_0^{(n)}(x)$ is not normalizable, the two hamiltonians are strictly isospectral, that is:

$$E_k^{(n)} = \widetilde{E}_k^{(n)}, \ k \geq 0, \tag{10}$$

and their eigenstates are related by:

$$\begin{cases} \psi_k^{(n)}(x) = \frac{1}{\sqrt{E_k^{(n)}}} A^{(n)}\widetilde{\psi}_k^{(n)}(x) \\ \widetilde{\psi}_k^{(n)}(x) = \frac{1}{\sqrt{E_k^{(n)}}} A^{(n)+}\psi_k^{(n)}(x) \end{cases}. \tag{11}$$

Suppose that the potential considered satisfies the following identity (this is the case of the harmonic and isotonic potentials):

$$-V(ix) = V(x) + \delta. \tag{12}$$

We then have:

$$V^{(n)}(x) = V(x) + \delta + E_n. \tag{13}$$

The spectrum of $H^{(n)}$ is:

$$
\begin{cases}
E_k^{(n)} = E_k + E_n + \delta \\
\psi_k^{(n)}(x) = \psi_k(x) \sim \exp\left(-\int w_k(x)dx\right)
\end{cases} , \quad k \geq 0.
\tag{14}
$$

As for the spectrum of $\widetilde{H}^{(n)}$, it is either:

$$
\begin{cases}
\widetilde{E}_k^{(n)} = E_k + E_n + \delta \\
\widetilde{\psi}_k^{(n)}(x) = \psi_k(x)
\end{cases} , \quad k \geq 0,
\tag{15}
$$

in the strictly isospectral case or:

$$
\begin{cases}
\widetilde{E}_0^{(n)} = 0 \\
\widetilde{E}_{k+1}^{(n)} = E_k + E_n + \delta
\end{cases} , \quad k \geq 0,
\tag{16}
$$

with:

$$
\begin{cases}
\widetilde{\psi}_0^{(n)}(x) \sim \exp\left(-\int v_n(x)dx\right) \\
\widetilde{\psi}_{k+1}^{(n)}(x) = (E_k + E_n + \delta)^{-1/2} A^{(n)+}\psi_k(x)
\end{cases} , \quad k \geq 0,
\tag{17}
$$

in the almost isospectral case.

We can illustrate this general scheme with two fundamental examples.

B. Harmonic oscillator

Consider the harmonic oscillator with zero ground-state energy:

$$
V(x) = \frac{\omega^2}{4}x^2 - \frac{\omega}{2}.
\tag{18}
$$

Its spectrum is well known:

$$
\begin{cases}
E_n = n\omega \\
\psi_n(x) \sim H_n(\omega x/2)\exp\left(-\omega x^2/4\right)
\end{cases}
\tag{19}
$$

and the corresponding RS functions $w_n(x)$ can be written as terminating continued fractions [11]. We then obtain for its "spatially Wick rotated" image $v_n(x) = -iw_n(ix)$:

$$
v_n(x) = \frac{\omega}{2}x + \frac{n\omega}{\omega x+} \nearrow \ldots \nearrow \frac{(n-j+1)\omega}{\omega x+} \nearrow \ldots \nearrow \frac{1}{x}.
\tag{20}
$$

Clearly $v_n(x)$ does not present any singularity on the positive real half line. The recurrence between the RS functions [11] gives:

$$
v_n(x) = v_0(x) + \frac{E_n}{v_0(x) + v_{n-1}(x)}
\tag{21}
$$

and $v_n(x)$ has the same odd parity as $v_0(x)$. Then $v_n(x)$ is regular on all $\mathbb{R}$. Therefore the normalizability of $v_n(x)$ is then ensured since the asymptotic behaviour at ∞ of $v_n(x)$ is that of $v_0(x)$.

We have also:

$$
-V(ix) = V(x) + \omega,
\tag{22}
$$

that is, $\delta = \omega$ and:

$$V^{(n)}(x) = V(x) + (n+1)\,\omega. \tag{23}$$

The spectrum of $H^{(n)}$ is then:

$$\begin{cases} E_k^{(n)} = (k+n+1)\,\omega \\ \psi_k^{(n)}(x) = \psi_k(x) \end{cases}, \quad k \geq 0. \tag{24}$$

Its SUSY partner $\widetilde{H}^{(n)}$ has the following associated potential:

$$\widetilde{V}^{(n)}(x) = 2v_n^2(x) - \frac{\omega^2}{4}x^2 - (n+1)\,\omega \tag{25}$$

and constitutes a regular rational extension of $V(x)$ the spectrum of which is completely determined. We have:

$$\begin{cases} \widetilde{E}_0^{(n)} = 0 \\ \widetilde{E}_{k+1}^{(n)} = (k+n+1)\,\omega \end{cases}, \quad k \geq 0 \tag{26}$$

and:

$$\begin{cases} \widetilde{\psi}_0^{(n)}(x) \sim \exp\left(-\int v_n(x)dx\right) \\ \widetilde{\psi}_{k+1}^{(n)}(x) = ((n+k+1)\,\omega)^{-1/2}\left(-\frac{d}{dx} + v_n(x)\right)\psi_k(x). \end{cases} \tag{27}$$

C.   Isotonic oscillator

The potential of the isotonic oscillator with zero ground-state energy is:

$$V(x) = \frac{\omega^2}{4}x^2 + \frac{l(l+1)}{x^2} - \omega\left(l + \frac{3}{2}\right), \quad x > 0. \tag{28}$$

Its spectrum is given by:

$$E_n = 2n\omega, \ \psi_n(x) \sim \exp\left(-\int w_n(x)\,dx\right), \tag{29}$$

where the $w_n(x)$ are known [11] and expressible as terminating continued fractions. This gives:

$$v_n(x) = \frac{\omega}{2}x + \frac{l+1}{x} + \frac{2n\omega}{\omega x + (2l+3)/x +} \ulcorner \dots \ulcorner \frac{2(n-j+1)\,\omega}{\omega x + (2(l+j)+1)/x +} \ulcorner \dots \ulcorner \frac{2\omega}{\omega x + (2(l+n)-1)/x}. \tag{30}$$

Clearly $v_n(x)$ does not present any singularity on the positive real half line. It has to be noticed that, since

$$v_0(x) = \frac{\omega}{2}x + \frac{l+1}{x}, \tag{31}$$

the term $(l+1)/x$ which then appears in every $v_n(x)$, induces a nonnormalizable singularity at the origin for $\exp\left(-\int v_n(x)dx\right)$. For instance:

$$\exp\left(-\int v_0(x)dx\right) = \frac{1}{x^{l+1}}\exp\left(-\frac{\omega}{4}x^2\right). \tag{32}$$

We are consequently in the case of a strict isospectrality.

We also have:

$$-V(ix) = V(x) + 2\omega\left(l + \frac{3}{2}\right),$$ (33)

that is, $\delta = 2\omega\left(l + \frac{3}{2}\right)$ and:

$$V^{(n)}(x) = \frac{\omega^2}{4}x^2 + \frac{l(l+1)}{x^2} + 2\left(n + l + \frac{3}{2}\right)\omega.$$ (34)

The spectrum of $H^{(n)} = -d^2/dx^2 + V(x) + 2\left(n + l + \frac{3}{2}\right)\omega$ is:

$$\begin{cases} E_k^{(n)} = 2\left(k + n + l + \frac{3}{2}\right)\omega \\ \psi_k^{(n)}(x) = \psi_k(x) \sim \exp\left(-\int w_k(x)dx\right) \end{cases}, \ k \geq 0.$$ (35)

Its SUSY partner $\widetilde{H}^{(n)}$ has the following associated potential:

$$\widetilde{V}^{(n)}(x) = 2v_n^2(x) - \frac{\omega^2}{4}x^2 - \frac{l(l+1)}{x^2} - 2\left(n + l + \frac{3}{2}\right)\omega$$ (36)

and constitutes a regular rational extension of $V(x)$ the spectrum of which is completely determined. We have:

$$\widetilde{E}_k^{(n)} = 2\left(n + k + l + \frac{3}{2}\right)\omega, \ k \geq 0$$ (37)

and:

$$\widetilde{\psi}_k^{(n)}(x) = \frac{1}{\sqrt{2\left(n + k + l + \frac{3}{2}\right)\omega}}\left(-\frac{d}{dx} + v_n(x)\right)\psi_k(x).$$

III. SECOND CATEGORY POTENTIALS

As shown in [11], the translationally shape invariant potentials can be classified into two categories in which the potential can be brought into a harmonic or isotonic form respectively, using a change of variables which satisfy a constant coefficient Riccati equation. Consider the second category. If we except the isotonic case itself, which has been treated above, every potential of this category, with a zero ground-state energy $E_0 = 0$, is of the form [11]:

$$V_\pm(y; a) = \lambda\left(\lambda \mp \alpha\right)y^2 + \frac{\mu\left(\mu - \alpha\right)}{y^2} + \lambda_{0\pm}(a)$$ (38)

with $a = (\lambda, \mu)$, $\lambda_{0\pm}(a) = -\alpha\left(\lambda \pm \mu\right) - 2\lambda\mu$. The variable y is defined via:

$$\frac{dy(x)}{dx} = \alpha \pm \alpha y^2(x),$$ (39)

that is, $y(x) = \tan\left(\alpha x + \varphi_0\right)$,in the V_+ case (+ type) and $y(x) = \tanh\left(\alpha x + \varphi_0\right)$ or $y = \coth\left(\alpha x + \varphi_0\right)$, in the V_- case (− type).

The spectrum $(E_{n\pm}, w_{\pm n})$ of $H_\pm = -d^2/dx^2 + V_\pm(y; a)$ is known analytically [11]. We have for the energies:

$$E_{n\pm}(a) = \pm\left(\phi_{2,\pm}\left(a_n\right) - \phi_{2,\pm}\left(a\right)\right)$$

with $\phi_{2,\pm}(a) = (\lambda \pm \mu)^2$ and $a_n = (\lambda_n, \mu_n) = (\lambda \pm n\alpha, \mu + n\alpha)$.

As for the RS functions, they are given by:

$$w_{n,\pm}(y,a) = \lambda y - \frac{\mu}{y} \mp \frac{\phi_{2,\pm}(a_n) - \phi_{2,\pm}(a)}{(\lambda + \lambda_1)y - (\mu + \mu_1)/y \mp} \mathbin{\rightarrow} \ldots \tag{40}$$

$$\mathbin{\rightarrow} \frac{\phi_{2,\pm}(a_n) - \phi_{2,\pm}(a_{j-1})}{(\lambda_{j-1} + \lambda_j)y - (\mu_{j-1} + \mu_j)/y \mp} \mathbin{\rightarrow} \ldots$$

$$\mathbin{\rightarrow} \frac{\phi_{2,\pm}(a_n) - \phi_{2,\pm}(a_{n-1})}{(\lambda_{n-1} + \lambda_n)y - (\mu_{n-1} + \mu_n)/y}$$

and in particular:

$$w_{0,\pm}(y;a) = \lambda y - \frac{\mu}{y}. \tag{41}$$

The RS function $w_{\pm n}(y;a)$ associated the level $E_{\pm n}(a)$ satisfies:

$$-w'_{\pm n}(x;a) + w^2_{\pm n}(x;a) = V_{\pm}(x;a) - E_{\pm n}(a) \tag{42}$$

or:

$$-\alpha\left(1 \pm y^2\right)w'_{\pm n}(y;a) + w^2_{\pm n}(y;a) = V_{\pm}(y,a) - E_{\pm n}(a). \tag{43}$$

If we set $x \to ix$ and $y \to iy$, the change of variable Eq(39) is transformed into $dy/dx = \alpha \mp \alpha y^2$.

Define $v_{\mp n}(y;a) = -iw_{\pm n}(iy;a)$ Eq(43) becomes:

$$\alpha\left(1 \mp y^2\right)v'_{\mp n}(y;a) + v^2_{\mp n}(y;a) = V^{(n)}_{\mp}(y;a), \tag{44}$$

where:

$$\begin{aligned} V^{(n)}_{\mp}(y;a) &= E_{n\pm}(a) - V_{\pm}(iy;a) \\ &= \lambda_{-1}\left(\lambda_{-1} \pm \alpha\right)y^2 + \frac{\mu(\mu - \alpha)}{y^2} + E_{n\pm}(a) - \lambda_{0\pm}(a) \\ &= V_{\mp}(y;\overline{a}_{\mp}) + E_{n\pm}(a) - \left(\lambda_{0\pm}(a) + \lambda_{0\mp}(\overline{a})\right) \end{aligned} \tag{45}$$

with $\lambda_{-1} = \lambda \mp \alpha$ and $\overline{a} = (\lambda_{-1}, \mu)$. We recover, up to a constant, a second category potential but of the opposite of type and with a modified multiparameter.

The energy spectrum of $H^{(n)}_{\pm} = -d^2/dx^2 + V^{(n)}_{\pm}(y;a)$ is:

$$\begin{cases} E^{(n)}_{k\pm} = E_{n\pm}(a) + E_{k\mp}(\overline{a}) - \left(\lambda_{0\pm}(a) + \lambda_{0\mp}(\overline{a})\right) \\ \psi^{(n)}_{k\pm}(x) = \psi_{k\mp}(x) \end{cases}, \quad k \geq 0. \tag{46}$$

Eq(44) can be written as:

$$v'_{\pm n}(x;a) + v^2_{\pm n}(x;a) = V^{(n)}_{\pm}(x;a) \tag{47}$$

and $v_{\pm n}(x;a)$ is the superpotential associated with $V^{(n)}_{\pm}$. $H^{(n)}_{\pm}$ is therefore the SUSY partner of $\widetilde{H}^{(n)}_{\pm}$ given by:

$$\widetilde{H}^{(n)}_{\pm} = -\frac{d^2}{dx^2} + \widetilde{V}^{(n)}_{\pm}(x;a),$$

where:

$$\widetilde{V}_\pm^{(n)}(x;a) \;=\; V_\pm^{(n)}(x;a) - 2v'_{\pm n}(x;a) \tag{48}$$
$$=\; 2v_{\pm n}^2(x;a) - V_\pm^{(n)}(x;a),$$

that is:

$$\widetilde{V}_\pm^{(n)}(x;a) = 2v_{\pm n}^2(x;a) - V_\mp(y;\overline{a}) - E_{n\pm}(a) + \left(\lambda_{0\pm}(a) + \lambda_{0\mp}(\overline{a})\right). \tag{49}$$

The two hamiltonians $H_\pm^{(n)}$ and $\widetilde{H}_\pm^{(n)}$ are factorizable as:

$$\begin{cases} \widetilde{H}_\pm^{(n)} = A_\pm^{(n)+} A_\pm^{(n)} \\ H_\pm^{(n)} = A_\pm^{(n)} A_\pm^{(n)+}, \end{cases} \tag{50}$$

where $A_\pm^{(n)} = d/dx + v_{\pm n}(x;a) = \left(\alpha \mp \alpha y^2\right) d/dy + v_{\pm n}(y;a)$.

Both are positive definite and isospectral. They are even strictly isospectral as in the case of the isotonic oscillator. Indeed we have:

$$v_{\pm 0}(y;a) = \lambda y + \frac{\mu}{y}, \tag{51}$$

that is, $\exp\left(\int v_{\pm 0}(x;a)dx\right)$ and $\exp\left(\int v_{\pm n}(x;a)dx\right)$ present a nonnormalizable singularity at the zero of $y(x)$.

Consequently the energy spectrum of $\widetilde{H}_\pm^{(n)}$ is:

$$\widetilde{E}_{\pm k}^{(n)} = E_{\pm k}^{(n)} = E_{n\pm}(a) + E_{k\mp}(\overline{a}) - \left(\lambda_{0\pm}(a) + \lambda_{0\mp}(\overline{a})\right), \ \ k \geq 0, \tag{52}$$

and the corresponding eigenstates are given by:

$$\widetilde{\psi}_{\pm k}^{(n)}(x) = \frac{1}{\sqrt{E_{\pm k}^{(n)}}} A_\pm^{(n)+} \psi_{k\pm}^{(n)}(x) = \frac{1}{\sqrt{E_{\pm k}^{(n)}}} A_\pm^{(n)+} \psi_{k\mp}(x). \tag{53}$$

Then for every n the potential:

$$\widetilde{V}_\pm^{(n)}(x;a) = 2v_{\pm n}^2(x;a) - V_\mp(x;\overline{a}) - E_{n\pm}(a) + \left(\lambda_{0\pm}(a) + \lambda_{0\mp}(\overline{a})\right), \tag{54}$$

where:

$$v_{n,\pm}(y,a) \;=\; \lambda y + \frac{\mu}{y} \pm \frac{\phi_{2,\pm}(a_n) - \phi_{2,\pm}(a)}{(\lambda + \lambda_1)y + (\mu + \mu_1)/y\pm} \ \Gamma \ ... \tag{55}$$

$$\Gamma \ \frac{\phi_{2,\pm}(a_n) - \phi_{2,\pm}(a_{j-1})}{(\lambda_{j-1} + \lambda_j)y + (\mu_{j-1} + \mu_j)/y\pm} \ \Gamma \ ...$$

$$\Gamma \ \frac{\phi_{2,\pm}(a_n) - \phi_{2,\pm}(a_{n-1})}{(\lambda_{n-1} + \lambda_n)y + (\mu_{n-1} + \mu_n)/y},$$

constitutes a solvable regular rational extension of $V_\pm(x;a)$.

IV. CONCLUSION

We have shown that every translationally shape invariant potential of second category admits an infinite family of solvable regular rational extensions. All the members of this family are strictly isospectral to the original potential and the associated eigenstates are easily related to the initial ones by application of first order differential operators. The adaptation of the above scheme of extension to the case of shape invariant potentials of the first category is in progress.

V. ACKNOWLEDGMENTS

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